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On the Representation of Arbitrary Functions by Definite Integrals.

By Walter B. Ford.

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1. *Introduction.* Suppose there is given an arbitrary function $f(x)$ defined throughout the real interval $(-\pi, \pi)$ and let the Fourier series for $f(\alpha)$ be formed, it being understood that α represents any special value of x within the indicated interval. The sum of the first $n+1$ terms of the series may be put in the form

$$S_n(\alpha)=\int_{-\pi}^{\pi} f(x)\psi(n,x-\alpha)\,dx, \text{ where } \psi(n,x-\alpha)=\frac{\sin\frac{2n+1}{2}(x-\alpha)}{2\pi\sin\frac{x-\alpha}{2}}, \tag{1}$$

and it is a familiar fact that we then have, at least after suitable conditions have been placed upon $f(x)$ in the neighborhood of the point $x=\alpha$ *

$$\lim_{n=+\infty} S_n=\frac{f(\alpha-0)+f(\alpha+0)}{2}. \tag{2}$$

The facts just mentioned readily suggest a general problem as follows: Having given an arbitrary function $f(x)$ defined throughout any (finite) interval (a, b), consider the integral (analogous to (1))

$$I_n(\alpha)=\int_a^b f(x)\phi(n, x-\alpha)\,dx, \tag{3}$$

wherein $\phi(n, x-\alpha)$ now represents an undetermined function of n and $x-\alpha$, and let it be asked what conditions upon $\phi(n, x-\alpha)$ suffice to insure the following relation analogous to (2):

$$\lim_{n=+\infty} I_n(\alpha)=\frac{f(\alpha-0)+f(\alpha+0)}{2}, \tag{4}$$

it being assumed as before that $f(x)$ satisfies suitable conditions in the neighborhood of the point $x=\alpha$.

The question thus raised was first considered and in part answered by Du Bois Reymond as early as 1875,† while in more recent years it has formed the subject of numerous and extensive researches of which those of Dini,[*] Hobson[†] and Lebesgue[‡] should be especially mentioned. For the most part, however, little attention appears to have been given to the actual determination of functions ϕ satisfying the conditions in question, the sole desideratum being the determination of the conditions themselves. In this connection it is proposed in the present paper to point out a noteworthy class of possible functions ϕ with especial emphasis on the corresponding integrals (3) to which they give rise. In particular, we shall show (§ 3) that to every convergent improper integral of the form

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* For example, whenever $f(x)$ has limited total fluctuation in the neighborhood in question.

† *Crelle,* Vol. LXXIX.

$$\int_0^\infty p(x)\,dx = k \neq 0$$

there corresponds an integral (3) having the property (4) provided merely that $p(x)$ is an even function of x, *i. e.*, $p(-x) = p(x)$, and $|xp(x)|$ remains less than a constant when considered for all values of x lying outside an arbitrarily small interval surrounding the origin.

2. We proceed at once to state and prove the following theorem:[§]

THEOREM I. *Let $F(x)$ be any single valued function of the real variable x defined for all finite values of x and satisfying the following three conditions:*

(a) $\lim\limits_{x=+\infty} F(x)$ *exists and* $= k \neq 0$,

(b) $F(-x) = -F(x)$,

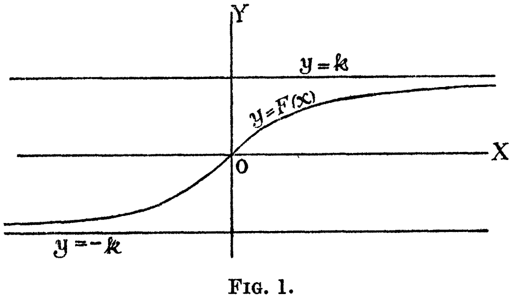

FIG. 1.

(c) *The derivative $F'(x)$ exists and is such that if we exclude the point $x=0$ by an arbitrarily small interval, we shall have for all remaining values of x, $|xF'(x)| < A =$ a constant.*

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[*] "Serie di Fourier" (Pisa, 1880), Chap. III.

[†] *Proc. London Math. Soc.*, Vol. VI (1908), pp. 349–395; *ibid.*, Vol. VII (1909), pp. 338–358. Both Dini and Hobson consider functions ϕ of the form $\phi(n, a, x-a)$ instead of the more limited form $\phi(n, x-a)$ mentioned above. We shall, however, confine our attention to the latter in this paper.

[‡] *Annales de Toulouse* (3), Vol. I (1909), pp. 25–128.

[§] The author's attention has recently been called by Dr. W. W. Küstermann to the fact that Pringsheim, in his lectures for 1910–11, developed a theorem closely allied to the one here stated, though it does not appear to have been given in published form.

Then, if $f(x)$ be an arbitrary function of the real variable x defined throughout the interval (a, b), we shall have for any special value $\alpha\,(a<\alpha<b)$

$$\lim_{n=+\infty} \frac{1}{2k}\int_a^b f(x)\,\frac{d}{dx}F[n(x-\alpha)]\,dx=\frac{f(\alpha-0)+f(\alpha+0)}{2},$$

provided merely that $f(x)$ satisfies suitable conditions (analogous to those under which the Fourier series for $f(\alpha)$ converges) in the neighborhood of the point $x=\alpha$.[*]

The proof of this theorem follows directly from the fact that if $f(x)$ satisfies the indicated conditions there exists the general relation

$$\lim_{n=+\infty} \int_a^b f(x)\phi(n, x-\alpha)\,dx=\frac{f(\alpha-0)+f(\alpha+0)}{2}$$

in case $\phi(n, t)$ is any function of the independent variables n and t satisfying the following three conditions, ε always denoting an arbitrarily small positive quantity:[†]

$$\text{(I)}\quad \lim_{n=+\infty}\int_0^t \phi(n, t)\,dt=\begin{cases} -\tfrac{1}{2} \text{ when } -\varepsilon<t<0,\\ +\tfrac{1}{2} \text{ when } 0<t<\varepsilon,\end{cases}$$

$$\text{(II)}\quad \left|\int_0^t \phi(n, t)\,dt\right|<c_1 \text{ when } -\varepsilon<t<\varepsilon, \quad c_1 \textit{ being a constant (independent of } n\text{)},$$

$$\text{(III)}\quad |\phi(n, t)|<c_2 \text{ when } \begin{Bmatrix} a-\alpha<t<-\varepsilon \text{ or}\\ \varepsilon<t<b-\alpha\end{Bmatrix}, \quad c_2 \textit{ being a constant (independent of } n\text{)}.$$

In fact, if $F(x)$ satisfies conditions (a), (b), (c), we have only to take

$$\phi(n, t)=\frac{1}{2k}\frac{d}{dt}F(nt) \tag{5}$$

in order to obtain a function $\phi(n, t)$ satisfying (I), (II) and (III), for we then have

$$\int_0^t \phi(n, t)\,dt=\frac{1}{2k}\Big[F(nt)\Big]_{t=0}^{t=t}, \tag{6}$$

and since by (b) we have $F(0)=0$, the right member of (6) reduces to $\frac{1}{2k}F(nt)$, and by (a) and (b) this is immediately seen to satisfy (I). Again, the assumption of the theorem that $F(x)$ is defined for all finite values of x

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[*] For the sake of definiteness, let us suppose that $f(x)$ has limited total fluctuation in the neighborhood in question, cf. Hobson's "Theory of Functions of a Real Variable," § 457. Let us also suppose throughout (in order to confine the attention to the simplest considerations and thus avoid exceptional aspects of the problem into which we do not care to enter in this paper) that $|f(x)|$ is finite and integrable in the Riemann sense over the interval (a, b).

[†] See for example Dini, *l. c.*, pp. 119–121; also, Ford, "Studies on Divergent Series and Summability" (1916), pp. 104–115.

and at the same time satisfies (a) and (b) shows at once, when employed in (6), that (II) is satisfied. Finally, (III) becomes satisfied as a result of (c) since (replacing nt by x) the second member of (5) may be put into the form

$$\frac{1}{2k}\frac{d}{dt}F(nt)=\frac{n}{2k}F'(x)=\frac{x}{2kt}F'(x),$$

and we are concerned in (III) only with values of t that do *not* include the value $t=0$.

Theorem I evidently enables one at once to construct an infinity of definite integrals (3) having the property (4). The simplest example of a function $F(x)$ such as described is perhaps that afforded by the branch of the transcendental curve $y=\text{arc}\tan x$ passing through the origin, the graph in this case having the general features indicated in the above figure, with $k=1$. Theorem I then gives the relation

$$\lim_{n=+\infty}\frac{n}{\pi}\int_a^b\frac{f(x)\,dx}{1+n^2(x-\alpha)^2}=\frac{f(\alpha-0)+f(\alpha+0)}{2}.^{*}$$

An example of an algebraic function $F(x)$ such as described in the theorem is easily supplied. Thus, consider the equation

$$(y^2-1)x+y=0. \tag{7}$$

The function y of x thus determined has a branch which passes through the origin and becomes asymptotic to the line $y=1$ when $x=+\infty$ and asymptotic to the line $y=-1$ when $x=-\infty$. The equation for this branch, as determined by solving (7) for y, is

$$y=\frac{-1+\sqrt{1+4x^2}}{2x}$$

and we readily see that this is a function of x which satisfies all the conditions of the $F(x)$ of the theorem, the value of k being 1. The resulting relation (3), as found by applying the theorem, becomes

$$\lim_{n=\infty}\frac{1}{4n^2}\int_a^b f(x)\left[\frac{\sqrt{1+4n^2(x-\alpha)^2}-1}{(x-\alpha)^2\sqrt{1+4n^2(x-\alpha)^2}}\right]dx=\frac{f(\alpha-0)+f(\alpha+0)}{2}.$$

Without multiplying examples, we pass at once to certain applications of the theorem in the study of improper definite integrals.

3. As an application of the result in § 2 we may readily establish the following:

THEOREM II. *Given any convergent improper definite integral of the form*

$$\int_0^\infty p(x)\,dx=k\neq 0,$$

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[*] This particular result has frequently been noted; cf. for example, Dini, *l. c.*, p. 36.

wherein (a) $p(x)$ *is an even function of* x, *i. e.*, $p(-x)=p(x)$ *and* (b) *the expression* $|xp(x)|$ *remains less than a constant when considered for all values of* x *lying outside an arbitrarily small interval surrounding the point* $x=0$.

Then, if $f(x)$ *be an arbitrary function of the real variable* x *defined throughout the interval* (a, b), *we shall have for any special value* $\alpha (a<\alpha<b)$

$$\lim_{n=+\infty} \frac{n}{2k} \int_a^b f(x) p[n(x-\alpha)]\, dx = \frac{f(\alpha-0)+f(\alpha+0)}{2}$$

provided merely that $f(x)$ *satisfies suitable conditions** *in the neighborhood of the point* $x=\alpha$.

In fact, we see at once that under the existing conditions for $p(x)$ we have but to place

$$F(x)=\int_0^x p(x)\, dx$$

in order to obtain a function $F(x)$ which satisfies the (a), (b) and (c) of Theorem I, and with this $F(x)$ substituted in the conclusion of that theorem we arrive at the conclusion stated in Theorem II.

In order to illustrate the consequences of Theorem II, several well-known definite integrals of the type under consideration are given in the left column below, and to the right occur the corresponding integrals (3) having the property (4) to which they give rise:†

$$\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}; \qquad \frac{n}{\sqrt{\pi}} \int_a^b f(x)\, e^{-n^2(x-\alpha)^2}\, dx,$$

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}; \qquad \frac{1}{\pi} \int_a^b f(x)\, \frac{\sin n(x-\alpha)}{x-\alpha}\, dx,$$

$$\int_0^\infty \frac{dx}{1-x^4} = \frac{\pi}{4}; \qquad \frac{2n}{\pi} \int_0^b \frac{f(x)\, dx}{1-n^4(x-\alpha)^4},$$

$$\int_0^\infty \frac{x^{2q}\, dx}{1-x^{2r}} = \frac{\pi}{2r} \cot \frac{2q+1}{2r} \pi; \qquad r n^{2q+1} \tan \frac{2q+1}{2r} \pi \int_a^b f(x)\, \frac{(x-\alpha)^{2r}\, dx}{1-n^{2r}(x-\alpha)^{2r}}.$$

(*q, r positive integers with* $r>q\neq 0$)

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* See earlier footnote.

† It may be observed that the first integral in the right column is closely related to the one employed by Weierstrass in his well-known researches upon the representation of arbitrary functions by series of polynomials (cf. for example, Lebesgue, *l. c.*, pp. 25–28). In fact, the Weierstrass integral results from the one in question by merely placing $a=-\infty$, $b=+\infty$, thus involving an extension of the results embodied in the present paper to cover an infinite interval instead of the finite interval (a, b). Such extensions can doubtless be made, at least under suitable restrictions, in both Theorems I and II, but we shall leave the subject aside in this preliminary paper.

It may also be observed that the second integral in the right column is closely related to (1), though essentially different from the fact that the function ψ in (1) is in reality dependent upon the

4. We proceed to deduce two noteworthy theorems analogous to those of §§ 2 and 3, but making less restrictive conditions upon the functions $F(x)$ and $p(x)$ there involved. The theorems in question will be found to bear an analogy to the well-known fact that in the study of the Fourier series for $f(\alpha)$ if we demand merely that in the neighborhood *at the right* of the point $x=\alpha$ the function $f(x)$ shall satisfy suitable conditions (in the sense heretofore specified) then, instead of (2) we may write

$$\lim_{n=+\infty} \int_{\alpha}^{\pi} f(x)\,\psi(n, x-\alpha)\,dx = \frac{1}{2} f(\alpha+0),$$

the expression here appearing on the left being the same as (1) except for the substitution of α for the lower limit of integration. Likewise, it is well known that if we demand merely that in the neighborhood *at the left* of the point $x=\alpha$ the function $f(x)$ shall satisfy suitable conditions, we may write

$$\lim_{n=+\infty} \int_{-\pi}^{\alpha} f(x)\,\psi(n, x-\alpha)\,dx = \frac{1}{2} f(\alpha-0).$$

To the facts just noted from the study of Fourier series there correspond in the more general studies of Du Bois Reymond, Dini and others, as found in the researches cited above, the following:

(A) If conditions (I), (II), (III) of § 2 are satisfied only for the *positive* values of t there specified—*i. e.*, if (I) is satisfied when $0<t<\varepsilon$, (II) when $0<t<\varepsilon$ and (III) when $\varepsilon<t<b-\alpha$—then, whenever $f(x)$ satisfies suitable conditions at the right of the point $x=\alpha$, we may write

$$\lim_{n=+\infty} \int_{\alpha}^{b} f(x)\,\phi(n, x-\alpha)\,dx = \frac{1}{2} f(\alpha+0). \tag{8}$$

(B) If conditions (I), (II), (III) of § 2 are satisfied only for the *negative* values of t there specified—*i. e.*, if (I) is satisfied when $-\varepsilon<t<0$, (II) when $-\varepsilon<t<0$ and (III) when $a-\alpha<t<-\varepsilon$—then, whenever $f(x)$ satisfies suitable conditions at the left of the point $x=\alpha$, we may write

$$\lim_{n=\infty} \int_{a}^{\alpha} f(x)\,\phi(n, x-\alpha)\,dx = \frac{1}{2} f(\alpha-0). \tag{9}$$

Relations (8) and (9) together with their accompanying remarks lead precisely as in § 2 to the following two theorems, in the first of which it is to be noted that we are in no wise concerned with the behavior of $F(x)$ for

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limits of integration $-\pi$, π, while the functions ϕ of such integrals (3) as occur in Theorems I and II are independent of the limits a, b.

The third and fourth integrals of the left are selected as simple illustrations in which the integrand is algebraic. They are taken from Williamson's "Integral Calculus," pp. 140 and 142.

negative values of x, while in the second we are in no wise concerned with $F(x)$ for positive values of x, these facts being indicated by the dotting of certain portions of the accompanying diagrams:

THEOREM III. *Let $F(x)$ be any single valued function of the real variable x which, when considered for positive values only* (0 *included*) *of x satisfies the following three conditions:*

(a) $\lim\limits_{x=+\infty} F(x)$ *exists and* $=k \neq 0$,

(b) $F(0)=0$,

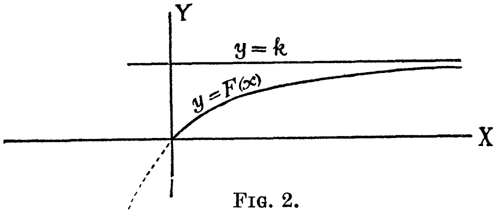

FIG. 2.

(c) *The derivative $F'(x)$ exists and is such that if we exclude the point $x=0$ by an arbitrarily small interval, we shall have for all remaining values of x, $|xF'_x(x)| < A =$ a constant.*

Then, if $f(x)$ be an arbitrary function of the real variable x defined throughout the interval (a, b), we shall have for any special value $\alpha (a<\alpha<b)$

$$\lim_{n=+\infty} \frac{1}{k} \int_{\alpha}^{b} f(x) \frac{d}{dx} F[n(x-\alpha)] = f(\alpha+0)$$

provided merely that $f(x)$ satisfies suitable conditions at the right of the point $x=\alpha$.

THEOREM IV. *Let $F(x)$ be any single valued function of the real variable x which, when considered for negative values only* (0 *included*) *of x satisfies the following three conditions:*

(a) $\lim\limits_{x=-\infty} F(x)$ *exists and* $=-k \neq 0$,

(b) $F(0)=0$,

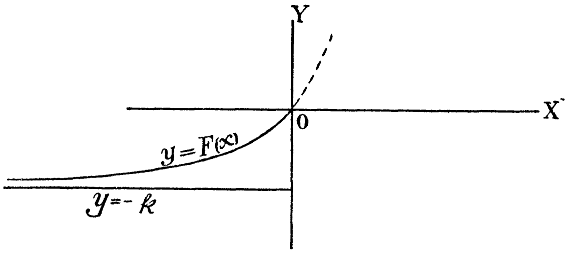

FIG. 3.

(c) *The derivative $F'(x)$ exists and is such that if we exclude the point $x=0$ by an arbitrarily small interval, we shall have for all remaining values of x, $|xF'(x)| < A =$ a constant.*

Then, if $f(x)$ be an arbitrary function of the real variable x defined throughout the interval (a, b), we shall have for any special value $\alpha (a<\alpha<b)$

$$\lim_{n=+\infty} \frac{1}{k} \int_a^\alpha f(x) \frac{d}{dx} F[n(x-\alpha)]dx = f(\alpha-0)$$

provided merely that $f(x)$ satisfies suitable conditions at the left of the point $x=0$.

By means of Theorems III and IV we may evidently arrive at other theorems analogous to that of § 3. These theorems are as follows, having been condensed, however, into one through the use of the ( ) to denote an alternative reading throughout:

THEOREM V. *Given any convergent improper integral of the form*

$$\int_0^\infty p(x)dx = k \neq 0 \quad \left(\int_{-\infty}^0 p(x)dx = k \neq 0\right),$$

wherein $|xp(x)|$ remains less than a constant when considered for all positive (negative) values of x lying outside an arbitrarily small interval to the right (left) of the point $x=0$.

Then, if $f(x)$ be an arbitrary function of the real variable x defined throughout the interval (a, b), we shall have for any special value $\alpha (a<\alpha<b)$

$$\lim_{n=+\infty} \frac{n}{k} \int_a^b f(x) p[n(x-\alpha)]dx = f(\alpha+0),$$

$$\left(\lim_{n=+\infty} \frac{n}{k} \int_a^\alpha f(x) p[n(x-\alpha)]dx = f(\alpha-0)\right)$$

*provided merely that $f(x)$ satisfies suitable conditions in the neighborhood to the right (left) of the point $x=\alpha$.**

Owing to the relative absence of restriction upon the $F(x)$ and $p(x)$ of Theorems III, IV, V, it is evident that illustrative examples are here supplied with even greater ease than was the case for Theorems I and II. The two following simple illustrations of Theorem V, wherein the same scheme as was indicated at the end of § 3 is carried out, will suffice:

$$\int_0^\infty x^m e^{-x} dx = \Gamma(m+1) \quad (m>-1); \qquad \lim_{n=+\infty} \frac{n^{m+1}}{\Gamma(m+1)} \int_a^b f(x)(x-\alpha)^m e^{-n(x-\alpha)} dx = f(\alpha+0),$$

$$\int_{-\infty}^0 \frac{dx}{(x-1)^2} = 1; \qquad \lim_{n=+\infty} n \int_a^\alpha \frac{f(x)dx}{[n(x-\alpha)-1]^2} = f(\alpha-0).$$

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* It may be observed at this point that Theorem I is a special consequence of Theorems III and IV obtained by applying III to the $F(x)$ of I when $x>0$ and IV to the $F(x)$ of I when $x<0$, then adding the two results and dividing by 2. Similarly, Theorem II is a special consequence of Theorem V.

5. The following general supplementary remarks may be noted in conclusion:

(1) Since the sum s_n of the first n terms of any convergent series approaches a limit as $n=+\infty$, we may frequently determine by means of the preceding theorems a certain integral (3) having the property (4) corresponding to a given convergent series. For example, consider the infinite geometric progression whose first term is a and whose common ratio is $r<1$. Then $s_n=\frac{a(1-r^n)}{1-r}$ and $\lim\limits_{n=+\infty} s_n=\frac{a}{1-r}$ so that the idea is readily suggested of taking as a possible function $F(x)$ the following:

$$F(x)=\frac{a(1-r^x)}{1-r}$$

and this in reality is at once seen to satisfy conditions (a), (b), (c) of Theorem III, the value of k being $\frac{a}{1-r}$. The theorem then gives

$$\lim_{n=+\infty} -n\,r^{-n\alpha}\log r\int_a^b f(x)\,r^{nx}dx=f(\alpha+0).$$

(2) The forms of representation discussed in the preceding §§ do not, strictly speaking, represent the given arbitrary function in terms of definite integrals, but rather in terms of the limits of such integrals as the parameter n increases to $+\infty$. However, it may be observed that the first member of (4) may always be expressed as an infinite series, viz.:

$$I_0(\alpha)+\sum_{n=0}^{\infty}[I_{n+1}(\alpha)-I_n(\alpha)]$$

and thus to every integral (3) such as obtained by any one of the preceding theorems there corresponds an actual representation of the arbitrary function $f(x)$ in series of definite integrals.

(3) An inspection of the proof of Theorem I shows directly that the theorem holds equally true when, instead of using the integral form given in the statement of the theorem, we employ the more general form

$$\int_a^b f(x)\,\frac{d}{dx}F[q(n)\,r(x-\alpha)]\,dx, \tag{10}$$

wherein $q(n)$ represents any function of n such that $\lim\limits_{n=+\infty} q(n)=+\infty$, while $r(x-\alpha)$ is such that $r(t)$ is positive for all positive values of t and negative for all negative values of t, and possesses a derivative $r'(t)$ at all points t. The possible integrals (3) having the property (4) and derivable by Theorem I thus become exceedingly varied. Similar generalizations are naturally possible in the statements of Theorems II–V. The nature of the results thus reached is indicated in the following table wherein to each theorem number in the first column there is given in the second column the form of expression (corresponding to (10)) to be used in the generalized theorem, it being understood always that $q(n)$ represents any function having the properties above indicated and that accents denote the derivative with respect to x:

$$\text{(II)} \quad \frac{q(n)r'(x-\alpha)}{2k}\int_a^b f(x)p[q(n)r(x-\alpha)]\,dx; \quad \left\{\begin{array}{l}r(t) \textit{ having the properties} \\ \quad \textit{above indicated.}\end{array}\right.$$

$$\text{(III)} \quad \frac{1}{k}\int_a^b f(x)F'[q(n)r(x-\alpha)]\,dx; \quad \left\{\begin{array}{l}r(t) \textit{ having for positive values of } t \\ \quad \textit{the properties above indicated, and} \\ \quad \textit{being such that } r(0)=0.\end{array}\right.$$

$$\text{(IV)} \quad \frac{1}{k}\int_a^a f(x)F'[q(n)r(x-\alpha)]\,dx; \quad \left\{\begin{array}{l}r(t) \textit{ having for negative values of } t \\ \quad \textit{the properties above indicated, and} \\ \quad \textit{being such that } r(0)=0.\end{array}\right.$$

$$\text{(V)} \quad \begin{array}{l}\dfrac{q(n)\,r'(x-\alpha)}{k}\displaystyle\int_a^b f(x)\,p[q(n)\,r(x-\alpha)]\,dx; \ r(t) \textit{ as in } \text{(III)}, \\ \left(\dfrac{q(n)\,r'(x-\alpha)}{k}\displaystyle\int_a^a f(x)\,p[q(n)\,r(x-\alpha)]\,dx; \ r(t) \textit{ as in } \text{(IV)}\right).\end{array}$$

Ann Arbor, *March*, 1915.